ACTA
Vol. 1 - N. 5

AN INFINITE INTEGRAL INVOLVING BESSEL FUNCTION AND SONINE'S POLYNOMIAL (*)

R. S. VARMA M. Sc.

(Christ Church College, Cawnpore, India)

SVMMARIVM. — Auctor, per usum symbolici calculi ab Heaviside introducti, valorem cuiusdam integralis dat, quod Besselianam functionem una cum Sonineo polynomio continet.

1. It has been shown by B. M. WILSON (¹) that the integral equation

$$f(x) = \lambda \int_0^\infty \sqrt{xy}\, J_m(xy)\, f(y)\, dy \tag{1.1}$$

wherein $R(m) > 1$, has characteristic numbers $\lambda = \pm 1$ and the corresponding solutions are

$$f(x) = e^{-\frac{1}{2}x^2} x^{m+\frac{1}{2}} T_m^n(x^2) \qquad [n = 0, 1, 2, \dots]$$

The object of this paper is to investigate an infinite integral involving BESSEL function and SONINE's polynomial which gives as a particular case the above integral equation.

(*) Nota inviata da Pierre Humbert S. C. e presentata dall'Accademico Pontificio Giuseppe Armellini, il 31 marzo 1937.

(¹) B. M. WILSON, *On an extension of Milne's integral equation*, «Messenger of Math.», 53 (1923-1924), 157-160.

2. We start with the following integral due to SONINE [1]:

$$\int_0^\infty e^{ky} e^{-y} y^m \, \mathrm{T}_m^n(y)\, dy = \frac{k^n}{n!\,(1-k)^{m+n+1}} \qquad [\mathrm{R}(k) < 1]$$

Putting $k = 1 - \alpha$, we obtain

$$\int_0^\infty e^{-\alpha y} y^m \, \mathrm{T}_m^n(y)\, dy = \frac{(1-\alpha)^n}{n!\,\alpha^{m+n+1}} \qquad [\mathrm{R}(\alpha) > 0] \tag{2.1}$$

Writing $\alpha = \beta + \dfrac{1}{p}$, this gives that

$$\int_0^\infty e^{-\beta y} e^{-\frac{y}{p}} \left(\frac{y}{p}\right)^m \mathrm{T}_m^n(y)\; dy$$

$$= \frac{p\,(\overline{1-\beta}\; p - 1)^n}{n!\,(p\beta + 1)^{m+n+1}}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r (1-\beta)^{n-r}}{r!\,(n-r)!} \quad \frac{p^{n-r+1}}{(p\beta+1)^{m+n+1}} \tag{2.2}$$

Consider now p as a symbolic operator. By virtue of the known operational image [2]

$$\left(\frac{y}{p}\right)^m e^{-\frac{y}{p}} \doteqdot (xy)^{\frac{1}{2}m} \;\; \mathrm{J}_m(2\sqrt{xy}) \;,$$

---

(1) N. SONINE, *Recherches sur les fonctions cylindriques et le développement des fonctions continues en series*, « Math. Annalen », 16 (1865), 1-80.

(2) B. VAN DER POL, *On the operational solution of linear differential equation and an investigation of the properties of these solutions*, « Phil. Mag. », VIII (1929), 861-898.

the original of the left hand side of [2.2] is

$$x^{\frac{1}{2}m} \int_0^{\infty} e^{-\beta y} \, y^{\frac{1}{2}m} \, J_m(2\sqrt{xy}) \, \overset{n}{T}_m(y) \, dy$$

To find the original of the right hand side, we write the integral [2.1] in the form

$$\int_0^{\infty} e^{-px} \, x^m \, \overset{n}{T}_m(x) \, dx = \frac{(1-p)^n}{n! \, p^{m+n+1}} \, ,$$

which gives that

$$[2.3] \qquad \frac{(1-p)^n}{n! \, p^{m+n}} \doteqdot x^m \, \overset{n}{T}_m(x) \qquad \left[\mathrm{R}(m) > -1\right]$$

Now we know from CARSON (1) that, if

$$\varphi(p) \doteqdot f(x) \, ,$$

then

$$[2.4] \qquad \varphi\left(\frac{p}{s}\right) \doteqdot f(sx) \, , \qquad (s = \text{const} > 0)$$

and

$$[2.5] \qquad \frac{p}{p+\alpha} \, \varphi(p+\alpha) \doteqdot e^{-\alpha x} f(x) \, ,$$

---

(1) CARSON, *Electric Circuit Theory and the operational calculus* (Mc Graw Hill, New York, 1926).

Applying first [2.5] and then [2.4] to [2.3], we get

$$\frac{(-1)^n}{n!} \quad \frac{(p\beta)^{n+1}}{(p\beta+1)^{m+n+1}} \doteqdot e^{-\frac{y}{\beta}} \left(\frac{y}{\beta}\right)^m \mathrm{T}_m^n\left(\frac{y}{\beta}\right)$$

The original of the right hand side in [2.2] is, hence,

$$(-1)^n \sum_{r=0}^{n} \frac{(1-\beta)^{n-r}}{r!\,\beta^{n-r+1}} \; e^{-\frac{x}{\beta}} \left(\frac{x}{\beta}\right)^{m+r} \mathrm{T}_{m+r}^{n-r}\left(\frac{x}{\beta}\right)$$

It follows therefore that

$$\int_0^\infty e^{-\beta y} y^{\frac{1}{2}m} \mathrm{J}_m(2\sqrt{xy})\, \mathrm{T}^n(y)\, dy =$$

[2.6] $$= (-1)^n \sum_{r=0}^{n} \frac{(1-\beta)^{n-r}}{r!\,\beta^{m+n+1}} \; x^{\frac{1}{2}m+r} \, e^{-\frac{x}{\beta}} \, \mathrm{T}_{m+r}^{n-r}\left(\frac{x}{\beta}\right)$$

$$\left[\mathrm{R}(m) > -1\right]$$

3. To show that [2.6] gives as a particular case [1.1] we shall require the following

Lemma:

$$\sum_{r=0}^{n} \frac{x^r}{r!} \mathrm{T}_{m+r}^{n-r}(x) = \mathrm{T}_m^n(2x)$$

By the help of [2.3], we have

$$\sum_{r=0}^{n} \frac{x^{m+r}}{r!} \mathrm{T}_{m+r}^{n-r}(x) \doteqdot \sum_{r=0}^{n} \frac{(1-p)^{n-r}}{r!\,(n-r)!\,p^{m+n}}$$

$$= \frac{(2-p)^n}{n!\,p^{m+n}}$$

$$\doteqdot x^m \mathrm{T}_m^n(2x)$$

From thie the lemma at once follows.

Not for $\beta = \frac{1}{2}$, [2.6] reduces to

$$\int_0^\infty e^{-\frac{1}{2}y} y^{\frac{1}{2}m} \, J\,(2\sqrt{xy}\, T_m^n(y)\, dy$$

$$= (-1)^n 2^{m+1} x^{\frac{1}{2}m} e^{-2x} \sum_{r=0}^{n} \frac{(2x)^r}{r\,!} T_{m+r}^{n-r}(2x)$$

Using our lemma establishei above, it is easy to see that this result is equivalent to [1.1].